Linear Systems

Solution	Can have **no solution**, **one solution** or **infinitly many**. The solution is the **intersection**
Solution set	The set of all possible solutions
Consistency	A system is **consistent** if there is at least one solution otherwise it is **inconsistent**
Equivalent	Linear systems are **equivalent** if they have the same **solution set**
Row operations	**Replacement**, **interchange** and **scaling**
Row equivalent	If there is a sequence of **row operations** between two linear systems then the systems are **row equivalent**. Systems that are row equivalent has the same **solution set**.
Existence	If a system has a solution (i.e. **consistent**)
Uniqueness	Is the solution unique
Homogenous	A system is **homogenous** if it can be written in the form A**x** = 0
Trivial solution	If a system only has a the solution **x** = 0. A system with no **free variable** only have the trivial solution.
Non-trivial solution	A **nonzero** vector that satisfies A**x** = 0. Has free variable.

Inverser of a Matrix

C is invertible if **CA** = I^n and **AC** = I^n
If **A** is (2x2) then, A^{-1} =
$(A^{-1})^{-1} = A$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

Linear Transformations

Transformation/mapping	T(x) from R^n to R^m
Image	For x in R^n the vector T(x) in R^m is called the image
Range	The set of all **images** of the vectors in the **domain** of T(x)
Criterion for a transformation to be linear	1. T(u + v) = T(u) + T(v) 2. T(cU) = cT(U)
Standard Matrix	The matrix A for a linear transformation T, that satisfies T(x) = Ax for all x in R^n
Onto	A mapping T is said to be **onto** if each b in the codomain is the image of at least one x in the domain. Range = Codomain. Solution existance. ColA must match codomain.
One-to-one	If each b in the **codomain** is only the image **at most** one x in the **domain**. Solution Uniqueness.
	T is one-to-one if and only if the cols of A are **linearly independent**
Free variable?	If the system has a free variable, then the system is **not** one-to-one. I.e. the homogenous system only has the trivial solution
Pivot in every row?	Then T is **onto**

Linear Transformations (cont)

Pivot in every column?	Then T is **one-to-one**

To determine whether a vector **c** is in the range of a **T**. **Solution**: Let T(x) = Ax. Solve the matrix equation Ax = c. If the system is **consistent**, then c is in the range of T.

The Invertible Matrix Theorem

The following statements are equivalent i.e. either they are all true or all false. Let A be a (*n*x*n*) matrix
A is an invertible matrix.
A is row equivalent to the $n \times n$ identity matrix
A has n pivot positions.
The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
The columns of A form a linearly independent set
The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in R^n
The columns of A span R^n
The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps R^n onto R^n
There is an $n \times n$ matrix C such that ☐☐$A = I$.
There is an $n \times n$ matrix D such that ☐☐$D = I$.
A^T is an invertible matrix.
The columns of A form a basis of R^n
Col A = R^n
Dim Col A = n
Rank A = n
Nul A = {0}
Dim Nul A = 0
The number 0 is not an eigenvalue of A
The determinant of A is not 0

Elementary Matrices

Elementary Matrix Is obtained by performing a single elementary row operation on an **identity matrix**

Each elementary matrix **E** is **invertible**

A nxn matrix A is invertible if and only if A is row equivalent to I^n.

$A = E^{-1}I^n$ and $A^{-1} = EI^n = I^n$

Row reduce the augmented matrix [A I] to [I A^{-1}]
NOTE If A is not row equivalent to I then A is not invertible

Linear Independence

A set of vectors are **linearly independent** if they cannot be created by any linear combinations of earlier vectors in the set.

If a set of vectors are **linear independent**, then the solution is **unique**

If the vector equeation c1*v1 + c2v*2 + ... + cp***v**p = 0 only has a **trivial solution** the set of vectors are **linearly independent**

Theorem: If a set contains more vectors than there are entries in each vector, then the set is **linearly dependent**

Theorem: If a set of vectors containt the zero vector, then the set is **linearly dependent**

Algebraic properties of a matrix

Algebraic properties of a matrix (cont)

$(AB)^T = B^T A^T$

For any scalar r, $(rA)^T = rA^T$

LU Factorization

Factorization of a matrix A is an equation that expresses A as a **product** of two or more matrices:
Synthesis: BC = A
Analysis: A= BC

Assumption: A is a $m \times n$ matrix that can be row reduced without **interchanges**

L: is a $m \times m$ unit lower triangular with 1's on the diagonal

U: is a $m \times n$ echelon form of A

U is equal to **E*A = U**, why **A = E^{1}U** = LU where **L = E^{-1}**

See figure ** for how to find L and U

Find **x** by first solving **Ly = b** and then solving **Ux = y**

Row Reduction and Echelon forms

Leading entry	A leading entry refers to the leftmost **non-zero** entry in a row
Echelon form	Row equivalent systems can be reduced into **several different** echelon forms
Reduced echelon form	A system is only row equivalent to **one** REF
Forward phase	Reducing an augmented matrix A into an **echelon form**
Backward phase	Reducing an augmented matrix A into a **reduced echelon form**
Basic variables	Variables in **pivot columns**.
Free variables	Variables that are not in **pivot columns**. When a system has a free variable the system is **consistent** but not **unique**

Subspaces of R^n

A **subspace** of R^n is any set H in R^n that has three properties:

- The zero vector is in H

- For each **u** and **v** in H, the sum **u + v** is in H

- For each **u** in *H* and each scalar *c*, the vector *c***u** is in *H*

Zero subspace is the set containing only the **zero vector** in R^n

Column space is the set of all linear combinations of the columns of A.

Null space (Nul A) is the set of all solutions of the equation **Ax = 0**

Basis for a subspace H is the set of **linearly independent** vectors that span H

In general, the **pivot columns** of A form a basis for col A

The number of vectors in any basis is **unique**. We call this number **dimension**

The **rank** of a matrix A, denoted by **rank** A, is the **dimension** of the **column space** of A

Determine whether b is in the **col A**.
Solution: b is only in col A if the equation **Ax = b** has a solution

Matrix and vector sum	A(**u** + **v**) = A**u** + A**v**
Matrix, vector and scalar	A(*c***u**) = *c*(A**u**)
Associ-ative law	A(BC) = (AB)C
Left distri-butive law	A (B + C) = AB + AC
Right distributive law	(B + C) = BA + BC
Scalar multiplic-ation	r(AB) = (rA)B = A(rB)
Identity matrix multi	$I^mA = A = AI^n$
Commute	If AB = BA then we say that A and B **commute** with each others
	$(A^T)^T = A$
	$(A + B)^T = A^T + B^T$

Algebraic properties of a vector

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u}$
$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$c(d\mathbf{u}) = (cd)\mathbf{u}$